

# A FORMULA FOR INSERTING POINT MASSES

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ABSTRACT. Let  $d\mu$  be a probability measure on the unit circle and  $d\nu$  be the measure formed by adding a pure point to  $d\mu$ . We give a formula for the Verblunsky coefficients of  $d\nu$ , based on a result of Simon.

## 1. INTRODUCTION

Suppose we have a probability measure  $d\mu$  on the unit circle  $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ . We define the inner product associated with  $d\mu$  and the norm on  $L^2(\partial\mathbb{D}, d\mu)$  respectively by

$$\langle f, g \rangle = \int_{\partial\mathbb{D}} \overline{f(e^{i\theta})} g(e^{i\theta}) d\mu(\theta) \quad (1.1)$$

$$\|f\|_{d\mu} = \left( \int_{\partial\mathbb{D}} |f(e^{i\theta})|^2 d\mu(\theta) \right)^{1/2} \quad (1.2)$$

The family of monic orthogonal polynomials associated with the measure  $d\mu$  is denoted as  $(\Phi_n(z, d\mu))_{n=0}^\infty$ , while the normalized family is denoted as  $(\varphi_n(z, d\mu))_{n=0}^\infty$ .

Let  $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\bar{z})}$  and  $\varphi_n^*(z) = \Phi_n^*(z)/\|\Phi_n\|$  be the reversed polynomials. Orthogonal polynomials obey the Szegő recursion relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \overline{\alpha_n}\Phi_n^*(z) \quad (1.3)$$

$\alpha_n$  is called the  $n^{\text{th}}$  Verblunsky coefficient. It is well known that there is a one-to-one correspondence between  $d\mu$  and  $(\alpha_j(d\mu))_{j=0}^\infty$  and that

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the Verblunsky coefficients carry much information about the family of orthogonal polynomials. For example,

$$\|\Phi_n\|^2 = \prod_{j=0}^{n-1} (1 - |\alpha_j|^2) \quad (1.4)$$

For a comprehensive introduction to the theory of orthogonal polynomials on the unit circle, the reader should refer to [4, 5], or the classic reference [6].

The result that we would like to present is the following

**Theorem 1.1.** *Suppose  $d\mu$  is a probability measure on the unit circle and  $0 < \gamma < 1$ . Let  $d\nu$  be the probability measure formed by adding a point mass  $\zeta = e^{i\omega} \in \partial\mathbb{D}$  to  $d\mu$  in the following manner*

$$d\nu = (1 - \gamma)d\mu + \gamma\delta_\omega \quad (1.5)$$

*Then the Verblunsky coefficients of  $d\nu$  are given by*

$$\alpha_n(d\nu) = \alpha_n + \frac{(1 - |\alpha_n|^2)^{1/2}}{(1 - \gamma)\gamma^{-1} + K_n(\zeta)} \overline{\varphi_{n+1}(\zeta)} \varphi_n^*(\zeta) \quad (1.6)$$

*where*

$$K_n(\zeta) = \sum_{j=0}^n |\varphi_j(\zeta)|^2 \quad (1.7)$$

*and all objects without the label  $(d\nu)$  are associated with the measure  $d\mu$ .*

The proof is based on a result obtained by Simon in the proof of Theorem 10.13.7 in [5] (See Theorem 2.1 below).

In fact, the following formula had been found by Geronimus [2]

$$\Phi_n(z, d\nu) = \Phi_n(z) - \frac{\Phi_n(\zeta) K_{n-1}(z, \zeta)}{(1 - \gamma)\gamma^{-1} + K_{n-1}(\zeta, \zeta)} \quad (1.8)$$

The formula for the real case was rediscovered by Nevai [3]. Later, the same formula for the unit circle case was rediscovered by Cachafeiro-Marcellan [1]. Unaware of Geronimus' result and the fact that Nevai's result also applies to the unit circle, Simon reconsidered this problem and proved formula (2.7) independently using a totally different method.

For applications of formula (1.6), the reader may refer to [7] and [8].

## 2. THE PROOF

First, we will prove a few lemmas.

**Lemma 2.1.** *Let  $\beta_{jk} = \langle \Phi_j(d\mu), \Phi_k(d\mu) \rangle_{d\nu}$ . Then*

$$\Phi_n(d\nu)(z) = \frac{1}{D^{(n-1)}} \begin{vmatrix} \beta_{00} & \beta_{01} & \cdots & \beta_{0n} \\ \vdots & & & \vdots \\ \beta_{n-10} & \beta_{n-11} & \cdots & \beta_{n-1n} \\ \Phi_0(d\mu) & \cdots & \cdots & \Phi_n(d\mu) \end{vmatrix} \quad (2.1)$$

where

$$D^{(n-1)} = \begin{vmatrix} \beta_{00} & \beta_{01} & \cdots & \beta_{0n-1} \\ \vdots & & & \vdots \\ \beta_{n-10} & \beta_{n-11} & \cdots & \beta_{n-1n-1} \end{vmatrix} \quad (2.2)$$

*Proof.* Let  $\tilde{\Phi}_n(d\nu)$  be the right hand side of (2.1). We observe that the inner product  $\langle \Phi_j(d\mu), \tilde{\Phi}_n(d\nu) \rangle_{d\nu}$  is zero for  $j = 0, 1, \dots, n-1$  as the last row and the  $j^{th}$  row of the determinant are the same. By expanding in minors, we see that the leading coefficient of  $\tilde{\Phi}_n(d\nu)$  in (2.1) is one. In other words,  $\tilde{\Phi}_n(d\nu)$  is an  $n^{th}$  degree monic polynomial which is orthogonal to  $1, z, \dots, z^{n-1}$  with respect to  $\langle, \rangle_{d\nu}$ , hence  $\tilde{\Phi}_n(d\nu)$  equals  $\Phi_n(d\nu)$ .  $\square$

**Lemma 2.2.** *Let  $C$  be the following  $(n+1) \times (n+1)$  matrix*

$$\begin{pmatrix} A & v \\ w & \beta \end{pmatrix} \quad (2.3)$$

where  $A$  is an  $n \times n$  matrix,  $\beta$  is in  $\mathbb{C}$ ,  $v$  is the column vector  $(v_0, v_1, \dots, v_{n-1})^T$  and  $w$  is the row vector  $(w_0, w_1, \dots, w_{n-1})$ . If  $\det(A) \neq 0$ , we have

$$\det(C) = \det(A) \left( \beta - \sum_{0 \leq j, k \leq n-1} w_k v_j (A^{-1})_{jk} \right) \quad (2.4)$$

*Proof.* We expand in minors, starting from the bottom row to get

$$\det(C) = \beta \det(A) + \sum_{0 \leq j, k \leq n-1} w_k v_j (-1)^{j+k+1} \det(\tilde{A}_{jk}) \quad (2.5)$$

where  $\tilde{A}_{jk}$  is the matrix  $A$  with the  $j^{th}$  row and  $k^{th}$  column removed.

By Cramer's rule, since  $\det(A) \neq 0$ ,

$$\tilde{A}_{jk} = (-1)^{j+k} \det(A) (A^{-1})_{jk} \quad (2.6)$$

proving Lemma 2.2.  $\square$

Next, we are going to prove the following formula by Simon [5]:

**Theorem 2.1.** *The Verblunsky coefficient of  $d\nu$  (as defined in (1.5)) is given by*

$$\alpha_n(d\nu) = \alpha_n - q_n^{-1} \overline{\gamma \varphi_{n+1}(\zeta)} \left( \sum_{j=0}^n \alpha_{j-1} \frac{\|\Phi_{n+1}\|}{\|\Phi_j\|} \varphi_j(\zeta) \right) \quad (2.7)$$

where

$$K_n(\zeta) = \sum_{j=0}^n |\varphi_j(\zeta)|^2 \quad (2.8)$$

$$q_n = (1 - \gamma) + \gamma K_n(\zeta) \quad (2.9)$$

$$\alpha_{-1} = -1 \quad (2.10)$$

and all objects without the label  $(d\nu)$  are associated with the measure  $d\mu$ .

*Proof.* Since  $\alpha_{n-1}(d\nu) = -\overline{\Phi_n(0, d\nu)}$  and  $\overline{\beta_{jk}} = \beta_{kj}$ , by Lemma 2.1,

$$\alpha_{n-1}(d\nu) = \frac{1}{D^{(n-1)}} \begin{vmatrix} \beta_{00} & \beta_{10} & \cdots & \beta_{n0} \\ \vdots & & & \vdots \\ \beta_{0n-1} & \beta_{1n-1} & \cdots & \beta_{nn-1} \\ -1 & \alpha_0 & \cdots & \alpha_{n-1} \end{vmatrix} \quad (2.11)$$

Let  $C$  be the matrix with entries as in the determinant in (2.11) above. It could be expressed as follows

$$C = \begin{pmatrix} A & v \\ w & \alpha_{n-1} \end{pmatrix} \quad (2.12)$$

where  $A$  is the  $n \times n$  matrix with entries  $A_{jk} = \beta_{kj}$ ,  $v$  is the column vector  $(\beta_{n0}, \dots, \beta_{nn-1})^T$  and  $w$  is the row vector  $(-1, \alpha_0, \dots, \alpha_{n-2})$ . Note that  $\det(A) = D^{(n-1)}$  and it is real as  $A$  is Hermitian.

Now we use Lemma 2.2 to compute  $\det(C)$ . To do that, we need to find out what  $A^{-1}$  is.

By the definition of  $\nu$ ,

$$A_{jk} = (1 - \gamma)\|\Phi_k\|^2\delta_{kj} + \gamma\overline{\Phi_k(\zeta)}\Phi_j(\zeta) = \|\Phi_k\|\|\Phi_j\|M_{jk} \quad (2.13)$$

where

$$M_{jk} = (1 - \gamma)\delta_{kj} + \gamma\overline{\varphi_k(\zeta)}\varphi_j(\zeta) \quad (2.14)$$

Observe that for any column vector  $x = (x_0, x_1, \dots, x_{n-1})^T$ ,

$$Mx = (1 - \gamma)x + \gamma \left( \sum_{j=0}^{n-1} \varphi_j(\zeta)x_j \right) (\varphi_0(\zeta), \varphi_1(\zeta), \dots, \varphi_{n-1}(\zeta))^T \quad (2.15)$$

Therefore, if  $P_\varphi$  denotes the orthogonal projection onto the space spanned by the vector  $\varphi = (\varphi_0(\zeta), \varphi_1(\zeta), \dots, \varphi_{n-1}(\zeta))$ , we can write

$$M = (1 - \gamma)\mathbf{1} + \gamma K_{n-1} P_\varphi \quad (2.16)$$

Hence, the inverse of  $M$  is

$$M^{-1} = (1 - \gamma)^{-1}(\mathbf{1} - P_\varphi) + ((1 - \gamma) + \gamma K_{n-1})^{-1} P_\varphi \quad (2.17)$$

and the inverse of  $A$  is

$$A^{-1} = D^{-1} M^{-1} D^{-1} \quad (2.18)$$

where  $D_{ij} = \|\Phi_i\|\delta_{ij}$ .

Recall that  $v = (\beta_{n0}, \beta_{n1}, \dots, \beta_{nn-1})^T$ , which is a multiple of  $\varphi$ . Therefore,

$$(A^{-1}v)_j = ((1 - \gamma) + \gamma K_{n-1})^{-1} \gamma \overline{\Phi_n(\zeta)} \|\Phi_j\|^{-1} \varphi_j(\zeta) \quad (2.19)$$

(2.19), (2.11) and Lemma 2.2 then imply

$$\alpha_{n-1}(d\nu) = \alpha_{n-1} - ((1 - \gamma) + \gamma K_{n-1})^{-1} \gamma \overline{\varphi_n(\zeta)} \left( \sum_{j=0}^{n-1} \alpha_{j-1} \frac{\|\Phi_n\|}{\|\Phi_j\|} \varphi_j(z_0) \right) \quad (2.20)$$

□

This concludes the proof of Theorem 2.1.

Now we are going to prove Theorem 1.1.

*Proof.* First, observe that  $\alpha_{j-1} = -\overline{\Phi_j(0)}$ . Therefore,  $\alpha_{j-1}/\|\Phi_j\| = -\overline{\varphi_j(0)}$ . Second, observe that  $\|\Phi_{n+1}\|$  is independent of  $j$  so it could be taken out from the summation. As a result, (2.7) in Theorem 2.1 becomes

$$\alpha_n(d\nu) = \alpha_n(d\mu) + q_n^{-1}\gamma \overline{\varphi_{n+1}(\zeta)} \|\Phi_{n+1}\| \left( \sum_{j=0}^n \overline{\varphi_j(0)} \varphi_j(\zeta) \right) \quad (2.21)$$

Then we use the Christoffel-Darboux formula, which states that for  $x, y \in \mathbb{C}$  with  $x\bar{y} \neq 1$ ,

$$(1 - \bar{x}y) \left( \sum_{j=0}^n \overline{\varphi_j(x)} \varphi_j(y) \right) = \overline{\varphi_n^*(x)} \varphi_n^*(y) - \bar{x}y \overline{\varphi_n(x)} \varphi_n(y) \quad (2.22)$$

Moreover, note that  $q_n^{-1}\gamma = ((1-\gamma)\gamma^{-1} + K_n(\zeta))^{-1}$ . Therefore, (2.21) could be simplified as follows

$$\alpha_n(d\nu) = \alpha_n + \frac{\overline{\varphi_{n+1}(\zeta)} \varphi_n^*(0) \varphi_n^*(\zeta)}{(1-\gamma)\gamma^{-1} + K_n(\zeta)} \|\Phi_{n+1}\| \quad (2.23)$$

Finally, observe that  $\varphi_n^*(0) = \|\Phi_n\|^{-1}$  and that by (1.4),  $\|\Phi_{n+1}\|/\|\Phi_n\| = (1 - |\alpha_n|^2)^{1/2}$ . This completes the proof.  $\square$

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